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THE STRUCTURE OF TRANSIENT BOUNDARY FLOW ALONG AN INCLINED PLANE IN A CONTINUOUSLY STRATIFIED MEDIUM†

A. V. KISTOVICH and YU. D. CHASHECHKIN

Moscow

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An exact solution is found to the unsteady problem of flow formation along an inclined plane, compensating diffusive transfer in a stationary infinitely deep continuously stratified fluid whose density depends only on the concentration of dissolved matter (salinity).

IN A STRATIFIED medium whose density is governed by an associated stable distribution of salinity (or temperature), a molecular flux of the stratifying agent is established. If it is horizontally inhomogeneous, then a necessary condition for the stability of the medium at rest is violated even when there are no destabilizing external factors. Such a phenomenon occurs near an impermeable inclined boundary at which the normal component of the gradient of the stratifying agent vanishes, so that the isohalines (isotherms), which in the undisturbed medium are situated horizontally, are distorted. The resulting pressure gradient in a homogeneous gravitational field accelerates the fluid layer along the boundary and forms a convective flux that compensates for the weakened molecular transfer near the boundary. The fact that the flow occurs does not depend on the shape of the boundary. The mathematical problem has its simplest form when an infinitely deep fluid is divided into two domains by an impermeable inclined plane. The flow ascends along the upper boundary and descends along the lower one.

This effect was first pointed out by Prandtl [1, p. 507] when analysing the nature of hill and valley winds in a thermally stratified atmosphere. A calculation [2, 3] was performed for a steady flow in the case of salt stratification, and asymptotic solutions were obtained [4] for the unsteady problem in the limits of short and long times.

1. STATEMENT OF THE PROBLEM

Consider an unbounded medium initially at rest, that is linearly stratified along the vertical direction, the density distribution being given by the relation

$$\rho = \rho_0 (1 + \beta S_i(z)), \quad \beta S_i(z) = -\sigma_0 z/\Lambda \tag{1.1}$$

Here z is the vertical coordinate pointing in the opposite direction to the gravitational force, Λ is the stratification length-scale, $\beta = (\partial p/\partial S)_{T,p}$ is the coefficient of salt compression, σ_0 is a dimensionless constant, S is the salinity, and $S_i(z)$ is the unperturbed salinity distribution.

At the initial instant the medium with density distribution (1.1) is at rest and there is no external perturbation. In the plane formulation we will seek the salinity distribution and the

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velocity field at subsequent times. The system of coordinates (ξ, η), $\xi \in (-\infty, +\infty)$, $\eta \in [0, +\infty)$ is attached to the plane (Fig. 1), and α is the angle of inclination of the plane to the horizontal.

The system of hydrodynamic equations in the Boussinesq approximation has the form

$$\begin{gathered} S_t' + uS_\xi' + vS_\eta' - \sigma_0 (u \sin\alpha + v\cos\alpha)/\Lambda = k (S_{\xi\xi}'' + S_{\eta\eta}'') \\ u_t' + uu_\xi' + vu_\eta' = -p_\xi' + \nu (u_{\xi\xi}'' + u_{\eta\eta}'') - Sg \sin\alpha \\ v_t' + uv_\xi' + vv_\eta' = -p_\eta' + \nu (v_{\xi\xi}'' + v_{\eta\eta}'') - Sg \cos\alpha \qquad u_\xi' + v_\eta' = 0 \end{gathered} \tag{1.2}$$

Here S is the perturbation of the initial salinity distribution normalized on β, u and v are the velocity components along ξ and η, respectively, g is the acceleration due to gravity, p is the pressure less the hydrostatic pressure reduced by ρ_0, and k and ν are the saline diffusivity and kinematic viscosity, respectively.

The no-slip condition for the velocity and the impermeability condition for the salinity are satisfied on the plane

$$(S_i + S)_\eta' = u = v = 0 \text{ when } \eta = 0$$

The initial conditions and the conditions at infinity are that the salinity, pressure and velocity perturbations vanish.

2. ANALYSIS OF POSSIBLE SOLUTIONS

Some properties of the form of the solution follow from an analysis of the problem.

Because the solution of system (1.2) has translational invariance with respect to shifts along the ξ axis, it can be represented in the form of a Fourier series expansion with respect to this variable

$$\begin{gathered} u = U_0 + \sum_{n=1}^{\infty} \{ U_n^c \cos\varphi_n (\xi, \eta) + U_n^s \sin\varphi_n (\xi, \eta) \} \\ v = \sum_{n=1}^{\infty} \{ V_n^c \cos\varphi_n (\xi, \eta) + V_n^s \sin\varphi_n (\xi, \eta) \} \\ S = S_0 + \sum_{n=1}^{\infty} \{ S_n^c \cos\varphi_n (\xi, \eta) + S_n^s \sin\varphi_n (\xi, \eta) \} \end{gathered} \tag{2.1}$$

Here $\varphi_n(\xi, \eta) = 2\pi n\xi/L$, where L is the translation period. Both L and the Fourier coefficients in (2.1) are functions of η, τ and α.

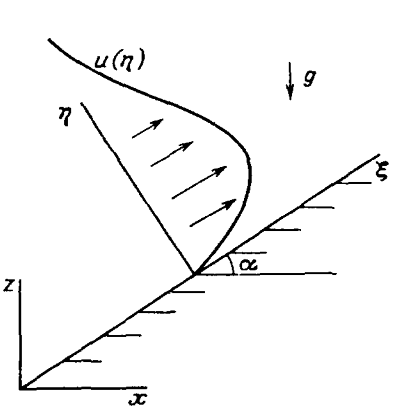

FIG. 1.

Investigation of the symmetry of the problem with respect to variation of the angle of inclination α leads to the relations

$$S_0(\alpha)=S_0(-\alpha),\ S_n^c(\alpha)=S_n^c(-\alpha),\ S_n^s(\alpha)=-S_n^s(-\alpha)$$
$$U_0(\alpha)=-U_0(-\alpha),\ U_n^c(\alpha)=-U_n^c(-\alpha),\ U_n^s(\alpha)=U_n^s(-\alpha) \tag{2.2}$$
$$V_n^c(\alpha)=V_n^c(-\alpha),\ V_n^s(\alpha)=-V_n^s(-\alpha)$$

By substituting (2.1) into (1.1) we can obtain a cumbersome but explicit system of equations and boundary conditions for the Fourier coefficients. The equations in this system are non-linear and it is therefore difficult to find the solution by some direct method. In this case, it is natural to use the method of successive approximations. The system of equations in the zeroth approximation has the form

$$S'_{0t}-\kappa S''_{0\eta\eta}=\sigma_0 U_0\Lambda^{-1}\sin\alpha,\ U'_{0t}-\nu U''_{0\eta\eta}=-S_0 g\sin\alpha \tag{2.3}$$

with boundary and initial conditions

$$S'_{0\eta}=\sigma_0\Lambda^{-1}\cos\alpha,\ U_0=0 \ \text{at}\ \eta=0$$
$$S_0=U_0=0\ \text{at}\ t=0$$

Here U_0 and S_0 are the Fourier coefficients of expansions (2.1).

In order to avoid non-uniformity of the solutions with respect to α, the functions U_0 and S_0 are expanded as Fourier series satisfying properties (2.2)

$$U_0=\sum_{n=1}^{\infty} H_n\sin(n\alpha),\ S_0=Q_0+\sum_{n=1}^{\infty} Q_n\cos(n\alpha) \tag{2.4}$$

where H_n and Q_n are functions of η and t.

As a result of substituting (2.4) into (2.3), an infinite system of equations is obtained for H_n and Q_n

$$Q'_{nt}-kQ''_{n\eta\eta}=\sigma_0(H_{n+1}-H_{n-1})/(2\Lambda),\ n=0,1,\ldots;\ H_0=H_{-1}=0$$
$$H'_{nt}-\nu H''_{n\eta\eta}=g(Q_{n-1}-Q_{n+1})/2,\ n=2,3,\ldots \tag{2.5}$$
$$H'_{1t}-\nu H''_{1\eta\eta}=g(2Q_0-Q_2)/2$$

Here

$$H_n=0,\ n=1,2,\ldots\ \text{at}\ \eta=0,$$
$$Q'_{1\eta}=\sigma_0/\Lambda,\ Q'_{n\eta}=0,\ n\neq 1\ \text{at}\ \eta=0$$
$$H_n=Q_n=0\ \text{at}\ t=0$$

It follows from (2.5) and the initial and boundary conditions that

$$Q_{2n}=H_{2n+1}=0,\ n=0,1,\ldots$$

As a result, expansion (2.4) acquires the form

$$U_0=\sum_{n=1}^{\infty} M_n\sin 2n\alpha,\ S_0=\sum_{n=2}^{\infty} K_n\cos(2n-1)\alpha$$

Here it is convenient to change to dimensionless physical coordinates, normalizing the time to the buoyancy frequency $N=(g/\Lambda)^{1/2}$, the coordinates to the length-scale $(\nu/N)^{1/2}$, the velocity to $(\Lambda g)^{1/2}$, and the salinity to $\sigma_0/2$. Moreover, we introduce two dimensionless parameters $A=-\sigma_0/4$ and $\epsilon=k/\nu$. Then for the functions M_n and K_n we have the equations

$$\begin{aligned}
&K'_{nt}-\epsilon K''_{n\eta\eta}=M_n-M_{n-1}\\
&M'_{nt}-M''_{n\eta\eta}=A\,(K_n-K_{n+1}),\ n=1,2,\ldots,\ M_0=0\\
&M_n=K'_{2\eta}=\ldots=K'_{n\eta}=\ldots=0,\ K'_{1\eta}=2\sqrt{\nu/(N\Lambda^2)}\quad \text{at}\quad \eta=0
\end{aligned}\tag{2.6}$$

In order to carry out an iterative procedure for finding the coefficients $\{M_i\}$, $\{K_i\}$, it is necessary to make a change of variables in order to separate the variables.

3. SOLUTION OF THE PROBLEM IN THE ZEROTH APPROXIMATION

At the first stage of the solution we change from the variables $\{M_i\}$, $\{K_i\}$, to the variables $\{\mu_i\}$, $\{\kappa_i\}$ using the rules

$$\begin{aligned}
&\kappa_1=K_1+K_2+K_3+\ldots,\quad \mu_1=M_1+2M_2+3M_3+\ldots\\
&\kappa_2=K_2+3K_3+6K_4+\ldots,\quad \mu_2=M_2+4M_3+10M_4+\ldots\\
&\kappa_3=K_3+5K_4+15K_5+\ldots,\quad \mu_3=M_3+6M_4+21M_5+\ldots,
\end{aligned}\tag{3.1}$$

The coefficients in these infinite expansions are obtained using Pascal's triangle, see Fig. 2. As a result system (2.6) takes the form

$$\begin{aligned}
&\kappa'_{nt}-\epsilon\kappa''_{n\eta\eta}=\mu_{n-1}\\
&\mu'_{nt}-\mu''_{n\eta\eta}=A\kappa_n,\ n=1,2,\ldots;\ \mu_0=0;\\
&\mu_n=\kappa'_{2\eta}=\ldots=\kappa'_{n\eta}=\ldots=0,\ \kappa'_{1\eta}=2\sqrt{\nu/(N\Lambda^2)}\quad \text{at}\quad \eta=0
\end{aligned}\tag{3.2}$$

Sequential integration of (3.2) using Green's functions gives rise to considerable technical difficulties. Hence to obtain an explicit form of the solution we perform a second stage, in which we determine the set of infinitesimal operators admitted by system (3.2)

$$\begin{aligned}
&\omega_1=\sum_{n=1}^{\infty}(\kappa_n\partial_{\kappa_n}+\mu_n\partial_{\mu_n}),\\
&\omega_2=2t\partial_t+\eta\partial_\eta+2\sum_{n=1}^{\infty}[2(n-1)\kappa_n\partial_{\kappa_n}+(2n-1)\mu_n\partial_{\mu_n}]\\
&\{\ v_m=\sum_{n=1}^{\infty}(\kappa_n\partial_{\kappa_{n+m}}+\mu_n\partial_{\mu_{n+m}}),\ m=1,2,\ldots\ \}
\end{aligned}$$

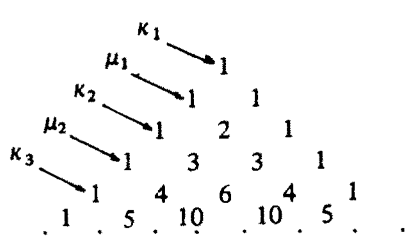

FIG. 2.

We then determine the group invariants of the linear combination

$$\omega_1+\omega_2+\sum_{m=1}^{\infty} a_m v_m,$$

as a result of which the solution of system (3.2) is represented in the form

$$\kappa_n=R_n(y)t^{(4n-3)/2},\ \mu_n=T_n(y)t^{(4n-1)/2};\ y=\eta/t^{1/2} \tag{3.3}$$

Substituting (3.3) into (3.2) we obtain a system of ordinary differential equations whose solution is described by the expressions

$$\begin{aligned}
&R_n=\sum_{m=1}^{n}\alpha_m^n U\left(q-\frac{1}{2},\frac{y}{\sqrt{2\epsilon}}\right)+\sum_{m=1}^{n-1}\beta_m^n U\left(q-\frac{1}{2},\frac{y}{\sqrt{2}}\right)\\
&T_n=\sum_{m=1}^{n}\gamma_m^n U\left(q+\frac{3}{2},\frac{y}{\sqrt{2\epsilon}}\right)+\sum_{m=1}^{n}\delta_m^n U\left(q+\frac{3}{2},\frac{y}{\sqrt{2}}\right)\\
&\left(q=4n-2m,\ U\left(n+\frac{1}{2},x\right)=\frac{1}{n!}\int_x^{\infty}(\xi-x)^n\exp(-\xi^{3/2})\,d\xi\right)
\end{aligned} \tag{3.4}$$

The coefficients α_m^n, β_m^n, γ_m^n and δ_m^n are given by the recurrence relations

$$\begin{aligned}
&\alpha_1^1=-4(k/\pi)^{1/2}(g\Lambda^3)^{-1/4}\\
&m\alpha_{m+1}^n=\gamma_m^{n-1},\ m\delta_{m+1}^n=A\beta_m^n,\ m\gamma_{m+1}^n+(\epsilon-1)\gamma_m^n/(2\epsilon)=A\alpha_m^n\\
&m=1,2,\ldots,n-1;\ n=2,3,\ldots\\
&m\beta_{m+1}^n+(1-\epsilon)\beta_m^n/2=\delta_m^{n-1},\ m=1,2,\ldots,n-2;\ n=3,4,\ldots\\
&(1-\epsilon)\beta_{n-1}^n/2=\delta_{n-1}^{n-1},\ n=2,3,\ldots\\
&(\epsilon-1)\gamma_n^n/(2\epsilon)=A\alpha_n^n,\ n=1,2,\ldots\\
&2^n\alpha_n^n+\sum_{m=1}^{n-1}2^m\{\alpha_m^n+\epsilon^{1/2}\beta_m^n\}\frac{\Gamma(n)}{\Gamma(2n-m)}=0,\ n=2,3,\ldots\\
&\sum_{m=1}^{n}\frac{2^m\{\gamma_m^n+\delta_m^n\}}{\Gamma(2n-m+3/2)}=0,\ n=1,2,\ldots
\end{aligned} \tag{3.5}$$

Then, after the functions R_n and T_n, and hence also κ_n, and μ_n, are determined, it is necessary, using relations (3.1), to determine K_n and M_n, which are the coefficients in the expansions for S_0 and U_0.

Reversing relations (3.1) we obtain

$$\begin{aligned}
&K_1=\kappa_1-\kappa_2+2\kappa_3-5\kappa_4+\ldots,\ M_1=\mu_1-2\mu_2+5\mu_3-\ldots\\
&K_2=\kappa_2-3\kappa_3+9\kappa_4-\ldots,\ M_2=\mu_2-4\mu_3+14\mu_3-\ldots\\
&K_3=\kappa_3-5\kappa_4+20\kappa_5-\ldots,\ M_3=\mu_3-6\mu_4+27\mu_5-\ldots
\end{aligned} \tag{3.6}$$

The coefficients in expansions (3.6) are obtained by means of a Pascal-type triangle in which the difference between the right and left elements of a row gives the value found between them in the row below (see Fig. 3).

Thus, a solution of system (2.3) is obtained which satisfies the appropriate initial and boundary conditions.

```
            ↓ K1
          M1 ↓
        K2 ↓ 1
      M2 ↓ 1    0
    K3 ↓ 1  −1
    ↓ 1  −2    0
    1  −3    2
 1  −4    5    0
 .  .  .  .  .  .
```

FIG. 3.

4. ANALYSIS OF THE SOLUTION

An important property of the solutions obtained is that they are also exact solutions of non-linear system (1.2) with the given boundary and initial conditions, which enables us to compare them with the results of [1–3] on a new basis.

The nature of the solution is governed by the structure of the function U from (3.4)

$$U\left(2n-\frac{1}{2},\ x\right)=e^{-x^2}\sum_{m=0}^{n} a_m^n x^{2m}+x\,\mathrm{erfc}\,(x)\sum_{m=0}^{n} b_m^n x^{2m}$$

where the explicit form of the coefficients a_m^n and b_m^n is governed by the form of the integral representation for U. Consequently, one can assert that the perturbation of the velocity field and density variation in the boundary flow are monotonic functions of time and do not have a stationary limit. The velocity distribution is a sign-constant function, unlike in [1–3] where the solution involves counterflows.

Analysis enables us to distinguish two time-dependent length-scales, describing the spatial distribution of the velocity and density (salinity) perturbations.

Far from the wall ($\infty>\eta>\sqrt{(2\nu t)}$) all the perturbations are small.

In the intermediate region ($\sqrt{(2\nu t)}>\eta>\sqrt{(2kt)}$) the perturbations reach a maximum value and the density variations are small. As in [1], this region may be referred to as the dynamical (or velocity) boundary layer, whose thickness is $\delta_p\sim\sqrt{(2\nu t)}$.

In the third region, which is directly adjacent to the plane at $\nu/k>1$, $\sqrt{(2kt)}>\eta\geqslant 0$, the salinity perturbation attains its maximum value. The velocity variation is weaker than in the second region. This part of the flow can be referred to as the concentration (or density) boundary layer, with thickness $\delta_p\sim\sqrt{(2kt)}$ [5].

The ratio of the characteristic sizes of these regions does not depend on time and is given by the Schmidt number $\delta_u/\delta_p=\sqrt{(\nu/k)}=\sqrt{(Sc)}$. The density boundary layer lies inside the dynamical layer when $Sc>1$ (metal salt solutions and sea water [6]). The dynamical layer is thinner than the density layer in the atmosphere, in metallic melts and in mercury ($Sc<1$). One can conjecture that in a multicomponent medium the number of such layers and the ratios of their length-scales are determined by the number and values of the transfer coefficients. (The steady-state solution [1–3] has a single length-scale because the distributions of the salinity and velocity perturbations are similar.) The exact solution agrees with the well-known solutions of the heat-conduction equation in a medium at rest when $\alpha=0$ and $\alpha=\pm\pi/2$. The velocity of the induced flow is a maximum when $\alpha=\pi/4$.

The first terms of solution (3.4)

$$S=\frac{\sigma_0}{\Lambda}\sqrt{\frac{kt}{\pi}}\left\{\sqrt{\pi\epsilon}\,y\,\mathrm{erfc}\left(\frac{y}{2\sqrt{\epsilon}}\right)-2\exp\left(-\frac{y^2}{4\epsilon}\right)\right\}\cos\alpha$$

$$U = \frac{2g\,t}{\Lambda(1-\epsilon)}\sqrt{\frac{kt}{\pi}}\left\{\frac{2}{3}(1+\frac{y^2}{4})\exp(\frac{y^2}{4}) - \frac{y\sqrt{\pi}}{2}(1+\frac{y^2}{6})\,\mathrm{erfc}(\frac{y}{2}) - \right.$$

$$\left. - \frac{2}{3}(1+\frac{y^2}{4\epsilon})\exp(-\frac{y^2}{4\epsilon}) + \frac{y}{2}\sqrt{\frac{\pi}{\epsilon}}(1+\frac{y^2}{6\epsilon})\,\mathrm{erfc}(\frac{y}{2\sqrt{\epsilon}})\right\}\sin 2\alpha$$

agree at short times with the one-component approximation of the asymptotic solution of [4]. The solution (3.4) obtained is analytic in all the physical parameters of the problem.

A special feature of the method used to construct the exact solution is that the approximate solutions containing a finite number of terms in (3.4) also satisfy the boundary and initial conditions exactly. This property is important because the existence of the flow is precisely due to the influence of the boundary. Because expansion (3.4) at a given instant of time converges fairly rapidly, for quantitative and qualitative analysis it is possible to retain just a few leading terms.

The unlimited increase of the solution with time implies the possibility of violating the Boussinesq approximation conditions at long times and the need to take into account the finite size of actual physical systems.

Comparison with high-resolution observations of stratified flows [5, 7, 8] shows that length-scale decoupling phenomena which substantially affect the structure and symmetry of complex flows in the immediate neighbourhood and at considerable distances from the body, are due to drifting of a decoupled boundary layer and induced by cutting the molecular background flux off by a resting or moving obstacle.

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